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Rings of Bonds in Graphs

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It is shown that every strict maximal ring of bonds of size greater than 3 is even.

1. INTRODUCTION

Throughout we consider finite graphs. In addition, all graphs are assumed to be undirected unless specified otherwise. The reader is referred to Bondy and Murty [1] for any notation and terminology not explained here.

Let G be a graph and $X, Y \subseteq V(G)$. As in [1] we denote by $[X, Y]$ the set of all edges joining a vertex of X and a vertex of Y. If $\bar{X} = V(G) - X$, then $[X, \bar{X}]$ is called an *edge cut* when both X and $\bar{X}$ are non-empty. We abbreviate $[X, \bar{X}]$ by $\delta(X)$ or $\delta(\bar{X})$. A minimal non-empty edge cut is a *bond*. If G is a directed graph and every edge is directed from the end in X, then $\delta(X)$ is a *directed edge cut*.

Let R be a set of bonds. If the edges of G can be oriented so that every bond of R is directed, then we say that R is *consistently orientable*. A cyclic sequence of bonds $R = (C_0, C_1, \ldots, C_{n-1})$ with $n \geqslant 3$ is a *ring* of bonds in the graph G if

(i) R is consistently orientable.

(ii) $C_i \cap C_j \neq \varnothing$ if and only if $i = j$, $i \equiv j + 1 \pmod n$ or $i \equiv j - 1 \pmod n$, and

(iii) no edge of G belongs to more than two bonds of R.

We note that (ii) implies (iii) except when $n = 3$.

For example, let G be the graph of Fig. 1. Then $(\delta(\{v_1\}), \delta(\{v_2\}), \delta(\{v_3\}), \delta(\{v_4, v_5\}))$ is a ring, as is evident from the given orientations of edges, but $(\delta(\{v_1\}), \delta(\{v_2\}), \delta(\{v_3\}), \delta(\{v_4\}), \delta(\{v_5\}))$ is not consistently orientable.

The *cardinality*, $|R|$, of R is the number of bonds in R. If $R = (C_0,$

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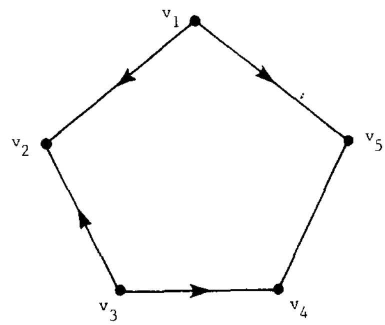

FIGURE 1

$C_1, \ldots, C_{n-1}$), then R is referred to as an *n-ring*. A ring is *odd* or *even* according to whether $|R|$ is odd or even.

The ring R is *strict* if there do not exist distinct bonds A, B, C satisfying the conditions $B \in R$, $C \in R$, $B \cap C = \varnothing$, $A \subseteq B \cup C$. Furthermore, R is *maximal* if there does not exist a ring $R' = (C_0', C_1', \ldots, C_{m-1}')$ in G such that $\bigcup_{k=0}^{m-1} C_k' \subseteq \bigcup_{l=0}^{n-1} C_l$ and $m > n$. We prove in this paper that every strict maximal ring with $n \geqslant 4$ is even.

The dual of this result for planar graphs together with the main theorem of [3] proves a characterization of planar graphs conjectured in [4]. Chernyak [2] has independently also used the main theorem of [3] to prove this conjecture. Thus we have the following theorem, where the reader is referred to [3] for the relevant definitions.

THEOREM 1. *A graph is planar if and only if it has no maximal strict odd ring of circuits.*

2. n-RINGS WITH $n \geqslant 4$

We begin by listing some elementary properties of bonds, each of which can be readily checked.

(1) Let G be a graph and let X be a non-empty proper subset of $V(G)$. The edge cut $\delta(X)$ is a bond if and only if $G[X]$ and $G[\bar{X}]$ are connected. (See Exercise 2.2.8 of [1].)

(2) Let $P = [S, \bar{S}]$ and $Q = [T, \bar{T}]$ be bonds of a connected graph. Then

(a) $\delta(S \cap T) \subseteq P \cup Q$;

(b) every bond from $P \cup Q$ distinct from P and Q contains an edge of $P - Q$ (and an edge of $Q - P$);

(c) $P \cap Q = [S \cap T, \bar{S} \cap \bar{T}] \cup [S \cap \bar{T}, \bar{S} \cap T]$;

(d) if P and Q are directed bonds, then at least one of $[S \cap T, \bar{S} \cap \bar{T}]$, $[S \cap \bar{T}, \bar{S} \cap T]$ is empty.

Let R be a strict maximal n-ring $(C_0, C_1, \ldots, C_{n-1})$ of G, where $n \geqslant 4$. We henceforth assume that G is oriented so that every bond in R is directed. For each $i \in \{0, 1, \ldots, n-1\}$ we choose $e_i \in C_i \cap C_{i+1}$. (Throughout this paper, all subscripts are to be read modulo n.) Let $C_i = \delta(A_i)$. Since $n \geqslant 4$ we have $e_{i-2} \notin C_i$. Therefore we may define D_i to be the element of $\{A_i, \bar{A}_i\}$ which does not contain vertices incident with e_{i-2}. For all i, we define $f(i) = 0$ if G is oriented so that every edge of C_i is directed toward the end in D_i and we let $f(i) = 1$ otherwise.

LEMMA 2. *We have $f(i) = f(i+1)$ if and only if*

$$C_i \cap C_{i+1} = [D_i \cap D_{i+1}, \bar{D}_i \cap \bar{D}_{i+1}].$$

On the other hand, $f(i) \neq f(i+1)$ if and only if

$$C_i \cap C_{i+1} = [D_i \cap \bar{D}_{i+1}, \bar{D}_i \cap D_{i+1}].$$

COROLLARY 3. *If $f(i) = f(i+1)$, then $[D_i \cap \bar{D}_{i+1}, \bar{D}_i \cap D_{i+1}] = \varnothing$. If $f(i) \neq f(i+1)$, then $[D_i \cap D_{i+1}, \bar{D}_i \cap \bar{D}_{i+1}] = \varnothing$.*

This corollary follows as a result of (2(d)).

LEMMA 4. *If an edge of C_j joins two vertices of D_i, then $j \equiv i \pm 1 \pmod{n}$.*

Proof. Suppose that e is an edge of C_j that joins two vertices of D_i, where $j \not\equiv i \pm 1 \pmod{n}$.

Suppose further that for every $k \notin \{i-1, i, i+1\}$, either $C_k \cap [D_i, D_i] = \varnothing$ or $C_k \cap [\bar{D}_i, \bar{D}_i] = \varnothing$. Recall that e_{i-2} joins two vertices of $\bar{D}_i$. Hence every edge of C_{i-2} joins vertices of $\bar{D}_i$ by assumption. Since $C_{i-2} \cap C_{i-3} \neq \varnothing$, the same is also true of C_{i-3} (if $n > 4$) and, by induction, of C_j, in contradiction to the existence of e. We conclude that for some $k \notin \{i-1, i, i+1\}$, $C_k \cap [D_i, D_i] \neq \varnothing$ and $C_k \cap [\bar{D}_i, \bar{D}_i] \neq \varnothing$.

By (2(a)), $\delta(D_i \cap D_k) \subseteq C_i \cup C_k$. Hence some subset of $\delta(D_i \cap D_k)$ is a bond C such that $C \subseteq C_i \cup C_k$. Since $C_k \cap [\bar{D}_i, \bar{D}_i] \neq \varnothing$ we have $C \neq C_k$. Suppose $C_i \cap [\bar{D}_k, \bar{D}_k] = \varnothing$. Then $[D_i \cap \bar{D}_k, \bar{D}_i \cap \bar{D}_k] = \varnothing$, so that $\delta(D_i \cap \bar{D}_k) = [D_i \cap D_k, D_i \cap \bar{D}_k] \subset C_k$ since $C_k \cap [\bar{D}_i, \bar{D}_i] \neq \varnothing$. On the other hand, $\delta(D_i \cap \bar{D}_k) \neq \varnothing$ because $D_i \cap \bar{D}_k \neq \varnothing$ and G is connected. These results contradict the fact that C_k is a bond. Hence $C_i \cap [\bar{D}_k, \bar{D}_k] \neq \varnothing$, and so $C \neq C_i$. Since $k \notin \{i-1, i, i+1\}$ we have $C_i \cap C_k = \varnothing$ and thus the strictness of R is contradicted. ∎

COROLLARY 5. *$C_j \subseteq [\bar{D}_i, \bar{D}_i]$ for all $j \notin \{i-1, i, i+1\}$.*

LEMMA 6. *For each i, $f(i) \neq f(i+1)$.*

Proof. Assume $f(i) = f(i+1)$ for some i, and without loss of generality, take $f(i) = 0$. We shall contradict the maximality of R by constructing a ring $\hat{R}$ in which C_i and C_{i+1} are replaced by three bonds: $\hat{C}_i$, $\hat{C}$ and $\hat{C}_{i+1}$.

Since $e_{i-1} \in C_i = [D_i, \bar{D}_i]$ and by Corollary 5, $e_{i-1} \in C_{i-1} \subseteq [\bar{D}_{i+1}, \bar{D}_{i+1}]$, then $e_{i-1} \in [D_i \cap \bar{D}_{i+1}, \bar{D}_i \cap \bar{D}_{i+1}]$. Let $\hat{D}_i$ be the vertex set of the component of $G[D_i \cap \bar{D}_{i+1}]$ containing an end of e_{i-1}.

We show that $\hat{C}_i = \delta(\hat{D}_i)$ is a bond. Let $\hat{G} = G[V(G) - \hat{D}_i]$. If L is a component of $\hat{G}$ containing vertices of D_{i+1}, then L contains all vertices of D_{i+1} since $G[D_{i+1}]$ is connected and $\hat{D}_i \subseteq \bar{D}_{i+1}$. Now $[D_i \cap D_{i+1}, \bar{D}_i] \neq \varnothing$ by Lemma 2, since otherwise $C_{i+1} \cap C_i = \varnothing$. Thus $G[\bar{D}_i]$ is a subgraph of L. Therefore any component M of $\hat{G}$ other than L contains only vertices of $D_i \cap \bar{D}_{i+1}$. Since G is connected, we have $[V(M), \hat{D}_i] \neq \varnothing$, so that the existence of M contradicts the definition of $\hat{D}_i$. Thus L is the only component of $\hat{G}$ and it follows that $\hat{C}_i$ is a bond.

Similarly, for any X such that $G[X]$ is a component of $G[\bar{D}_i \cap D_{i+1}]$, $\delta(X)$ is a bond. Let $\mathscr{D}$ be the union of the vertex sets of all such X for which $\delta(X) \cap C_{i+2} \neq \varnothing$.

By Lemma 2, $C_i \cap C_{i+1} = [D_i \cap D_{i+1}, \bar{D}_i \cap \bar{D}_{i+1}]$. Hence e_i has neither end in $D_i \cap \bar{D}_{i+1}$ and so $\hat{C}_i \subseteq (C_i \cap C_{i+1}) - \{e_i\}$. Thus $\hat{C}_i$ is distinct from C_i and C_{i+1} which implies that $\hat{C}_i$ contains an edge $\hat{e}_i \in C_{i+1} - C_i$. In particular, $\hat{e}_i \in [D_i \cap \bar{D}_{i+1}, D_i \cap D_{i+1}]$. Thus there is a component of $G[D_{i+1} \cap \bar{\mathscr{D}}]$ containing an end of $\hat{e}_i$. Call the vertex set of this component $\hat{D}$.

We now show that $\hat{C} = \delta(\hat{D})$ is a bond. Since $\hat{D} \subseteq D_{i+1}$ and C_{i+1} is a bond, $\bar{D}_{i+1}$ is contained in a single component Y of $G - \hat{D}$. Choose a component Z of $G - \hat{D}$. It suffices to show that $Z = Y$. If $V(Z) \cap \mathscr{D} \neq \varnothing$, then for some vertex set X such that $G[X]$ is a component of $G[\mathscr{D}]$, we have $X \subseteq V(Z)$. Because $G[X]$ is a component of $G[\bar{D}_i \cap D_{i+1}]$, any edge e of $\delta(X)$ must have an end $w \in D_i \cup \bar{D}_{i+1}$. If $w \in \bar{D}_{i+1}$, then $Z = Y$. Hence we assume that $\delta(X) = [X, D_i]$. But now we have the contradiction that $\delta(X) \cap C_{i+2} = \varnothing$ by Corollary 5. On the other hand, suppose $V(Z) \cap \mathscr{D} = \varnothing$. If $Z \neq Y$, then $V(Z) \subseteq D_{i+1} \cap \bar{\mathscr{D}}$. Thus $G[\hat{D}]$ cannot be a component of $G[D_{i+1} \cap \bar{\mathscr{D}}]$ because $[\hat{D}, V(Z)] \neq \varnothing$ by the fact that G is connected. Therefore $Z = Y$, so that $\hat{C}$ is a bond.

Suppose $e_{i+1} \in \hat{C}$. Then e_{i+1} joins a vertex $x \in D_{i+1} \cap \bar{\mathscr{D}}$ to a vertex $y \in \bar{D}_{i+1}$, since $e_{i+1} \in C_{i+1}$. But $e_{i+1} \in C_{i+2}$, and so $x \in \bar{D}_i$ by Corollary 5. Hence $e \in \delta(\bar{D}_i \cap D_{i+1})$ and since $e_{i+1} \in C_{i+2}$ the fact that $x \in \bar{\mathscr{D}}$ is contradicted. Thus $\hat{C} \subseteq C_i \cup C_{i+1} - \{e_{i-1}, e_{i+1}\}$. Hence $\hat{C}$ contains an edge $\hat{e} \in C_i - C_{i+1}$ by (2(b)). Then $\hat{e} \in [D_{i+1} \cap \bar{\mathscr{D}}, \mathscr{D}]$. Now we define $\hat{D}_{i+1}$ to be the vertex set of the component of $G[\mathscr{D}]$ containing an end of $\hat{e}$. Let $\hat{C}_{i+1} = \delta(\hat{D}_{i+1})$. We have already shown that $\hat{C}_{i+1}$ is a bond because $G[\hat{D}_{i+1}]$ is a component of $G[\mathscr{D}]$.

Define $\hat{R} = (C_0, C_1, \ldots, C_{i-1}, \hat{C}_i, \hat{C}, \hat{C}_{i+1}, C_{i+2}, \ldots, C_{n-1})$. We now aim to show that $\hat{R}$ is ring of bonds and thus contradict the maximality of R. First we reverse the direction of every edge of $\hat{C} \cup \{([D_i, D_i] \cup C_i) \cap ([D_{i+1}, D_{i+1}] \cup C_{i+1})\}$. Since $f(i) = 0$ each edge of C_i was directed toward the end in D_i in the original orientation of G.

Now we show that $\hat{R}$ is consistently orientable.

First we recall that $\hat{C} \subseteq \delta(D_{i+1} \cap \bar{\mathscr{D}})$, so that by Corollary 5 and the definition of a ring of bonds, $\hat{C}$ contains no edge of any bond in $R - \{C_i, C_{i+1}, C_{i+2}\}$. Hence it is clear that these bonds remain directed under the new orientation of G.

We have $\hat{C} \cap C_{i+2} = \varnothing$, for otherwise there is an edge of $\hat{C} \cap C_{i+2}$ in $[\bar{D}_i, \bar{D}_i]$, in contradiction to the definition of $\mathscr{D}$. Furthermore $C_{i+2} \cap [D_i, D_i] = \varnothing$ and $C_{i+2} \cap C_i = \varnothing$ so that C_{i+2} is a directed bond under the new orientation of G.

Now $\hat{C}_i \subseteq (C_i \cap [\bar{D}_{i+1}, \bar{D}_{i+1}]) \cup (C_{i+1} \cap [D_i, D_i])$ by Corollary 3. The edges of $C_i \cap [\bar{D}_{i+1}, \bar{D}_{i+1}]$ are directed towards $\hat{D}_i$ both before and after the reorientation of G. When G is reoriented the edges of $C_{i+1} \cap [D_i, D_i]$ are also directed towards $\hat{D}_i$. Hence $\hat{C}_i$ is a directed bond in the new orientation of G.

Next consider $\hat{C}$. Every edge in $\hat{C}$ has an end in $\hat{D} \cap D_{i+1}$ and every edge in $\hat{C} - C_{i+1}$ has an end in $\hat{D} \cap D_i$ by the definition of $\mathscr{D}$. Thus $\hat{C}$ is directed in both the old and new orientations of G.

In considering $\hat{C}_{i+1}$ we note that if there exists an edge $e' \in \hat{C}_{i+1} \cap [D_i, \hat{D}_{i+1}]$, then $e' \in [\bar{D}_i \cap \bar{D}_{i+1}, \hat{D}_{i+1} \cap \bar{D}_i \cap D_{i+1}]$, and hence is directed towards $\hat{D}_{i+1}$ in the new orientation. Hence $\hat{C}_{i+1}$ is a directed bond. Thus $\hat{R}$ is consistently orientable.

By construction we note that $e_{i-1} \in C_{i-1} \cap \hat{C}_i$, $\hat{e}_i \in \hat{C}_i \cap \hat{C}$, $\hat{e} \in \hat{C} \cap \hat{C}_{i+1}$ and $\hat{C}_{i+1} \cap C_{i+2} \neq \varnothing$. Hence consecutive bonds of $\hat{R}$ are not disjoint.

It remains to show that non-consecutive pairs of bonds in $\hat{R}$ are disjoint. Since $\hat{C}_i$, $\hat{C}$ and $\hat{C}_{i+1}$ are all subsets of $C_i \cup C_{i+1}$, they are disjoint from $C_{i+3}, C_{i+4}, \ldots, C_{i-2}$. This leaves only the five cases below to be considered.

(i) $C_{i-1} \cap \hat{C} = \varnothing$ (as has already been noted) since $C_{i-1} \subseteq [\bar{D}_{i+1}, \bar{D}_{i+1}]$ by Corollary 5.

(ii) $C_{i-1} \cap \hat{C}_{i+1} = \varnothing$ by the same argument.

(iii) $\hat{C}_i \cap \hat{C}_{i+1} = \varnothing$ by Corollary 3.

(iv) $\hat{C}_i \cap C_{i+2} = \varnothing$ since $C_{i+2} \subseteq [\bar{D}_i, \bar{D}_i]$ by Corollary 5.

(v) $\hat{C} \cap C_{i+2} = \varnothing$ as has already been shown.

Hence $\hat{R}$ is a ring of bonds. This conclusion contradicts the maximality of R and hence $f(i) \neq f(i+1)$ for each i. ∎

The main result of this paper is immediate.

THEOREM 7. *If $n \geqslant 4$, then every strict maximal n-ring of bonds is even.*

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